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# Quasi-Sampling Sets for Analytic Functions in a Cone

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We study analogues of sampling sets for analytic functions in cones of $\mathbf{C}^n$. Cartwright-type and Bernstein-type theorems, previously known only for functions of exponential type in $\mathbf{C}^n$, are extended to the case of functions of arbitrary order in a cone. 

## 1. INTRODUCTION

We use standard notations of multidimensional complex analysis.

Let $C$ be an open cone in $\mathbf{C}^n$ with vertex at the origin. By $H(C; \rho, \sigma)$ we denote the class of all functions $f$ holomorphic in $C$ and satisfying the estimate

$$\limsup_{|z| \to \infty,\, z \in C} \frac{\log |f(z)|}{|z|^{\rho}} \leqslant \sigma, \qquad |z|^2 = |z_1|^2 + \cdots + |z_n|^2.$$

By $H(C; \rho, \infty)$ we denote $\bigcup_{\sigma > 0} H(C; \rho, \sigma)$.

For entire functions we write simply $H(\rho, \sigma)$, $H(\rho, \infty)$, etc. Thus $H(1, \sigma)$ is the class of entire functions of exponential type not exceeding $\sigma$ in $\mathbf{C}^n$.

In 1937, M. Cartwright [C] proved the following theorem:



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THEOREM A. *The estimate*

$$\sup_{x \in \mathbf{R}} |f(x)| \leqslant C_\sigma \sup_{m \in \mathbf{Z}} |f(m)|$$

*holds for every function* $f \in H(1, \sigma)$ *with* $\sigma < \pi$. *Here the constant* $C_\sigma \in (0, \infty)$ *depends only on* $\sigma$.

Below we mention two Cartwright-type results for entire functions of several variables. We need some definitions first.

DEFINITION. Let $E$ and $F$ be subsets of $\mathbf{R}^n$, $E$ being measurable. The set $E$ is called *dense relative to* $F$, if for some positive constants $L$ and $\delta$ and every $x \in F$

$$|E \cap B(x, L)| \geqslant \delta.$$

Here $|A|$ denotes the Lebesque measure of a (measurable) set $A$, and $B(x, L)$ is the ball $\{ y \in \mathbf{R}^n : |x - y| < L \}$. The values of $L$ and $\delta$ are called the density characteristics (of $E$ relative to $F$).

Given $\eta \in (0, 1)$, denote by $C(\eta)$ the cone in the positive hyperoctant $\mathbf{R}^n_+$ defined by the relation

$$C(\eta) = \{ x \in \mathbf{R}^n_+ : \min_{j=1, \ldots, n} x_j > \eta \max_{j=1, \ldots, n} x_j \}.$$

THEOREM B. (a) *Let a set E be dense relative to* $\mathbf{R}^n$ *with density characteristics L and* $\delta$. *Then the estimate*

$$\sup_{x \in \mathbf{R}^n} |f(x)| \leqslant e^{C\sigma L^{n+1}/\delta} \sup_{x \in E} |f(x)|$$

*holds for some constant C, any* $\sigma \in (0, \infty)$ *and all* $f \in H(1, \sigma)$.

(b) *Let a set E be dense relative to* $\mathbf{R}^n_+$. *Then each entire function* $f \in H(1, \infty)$ *bounded on E is bounded on* $C(\eta)$ *for every* $\eta \in (0, 1)$. *Moreover, for each* $\sigma \in (0, \infty)$ *there exists a positive constant* $\Delta = \Delta(E, \sigma, \eta)$, *such that for each entire function* $f \in H(1, \sigma)$

$$\sup_{x \in C(\eta) \setminus B(0, R)} |f(x)| \leqslant \Delta \cdot \sup_{x \in E} |f(x)|$$

*for some* $R = R(f, \eta) < \infty$.

Statement (a) is due to B. Ya. Levin [Le]; statement (b) is due to the first author [L2].

DEFINITION. Let $\varepsilon$ be a positive number. A set $E \subset \mathbf{R}^n$ is called an *$\varepsilon$-net* for a set $F \subset \mathbf{R}^n$ if for every $x \in F$ there exists a point $y \in E$ such that

$$|x - y| < \varepsilon.$$

Note that $\varepsilon$-nets may be discrete sets.

THEOREM C [L1, L2]. (a) *If $\sigma\varepsilon < \pi/200$ and $E$ is an $\varepsilon$-net for $\mathbf{R}^n$, then the estimate*

$$\sup_{x \in \mathbf{R}^n} |f(x)| \leqslant \frac{1}{1 - \sigma\varepsilon} \sup_{x \in E} |f(x)|$$

*holds for every function $f \in H(1, \sigma)$.*

(b) *Let $E$ be an $\varepsilon$-net for $\mathbf{R}^n_+$, and let a number $\eta \in (0, 1]$ be given.*

*Then there exists a positive number $\sigma_0 = \sigma_0(n, \varepsilon, \eta) > 0$ such that for every $\sigma \in (0, \sigma_0)$, each function $f \in H(1, \sigma)$ bounded on $E$ is bounded on $C(\eta)$.*

*Moreover, if $\sup_{x \in E} |f(x)| > 0$, there is a positive constant $\Delta = \Delta(E, \sigma, \eta)$, such that*

$$\sup_{x \in C(\eta) \setminus B(0, R)} |f(x)| \leqslant \Delta \cdot \sup_{x \in E} |f(x)|$$

*for some $R = R(f, \eta) < \infty$.*

Let $B(\sigma)$ be the subclass of $H(1, \sigma)$ consisting of functions which are bounded on $\mathbf{R}^n$. A set $E \subset \mathbf{R}^n$ is called a *sampling set* for $B(\sigma)$ if

$$\sup_{x \in \mathbf{R}^n} |f(x)| \leqslant C \sup_{x \in E} |f(x)|, \qquad \forall f \in B(\sigma).$$

Investigation of such sets in one-dimensional case was initiated by A. Beurling [Beu]. The (a) parts of Theorems B and C give in particular sufficient conditions on a set $E$ to be a sampling set for $B(\sigma)$ in $\mathbf{C}^n$. The (b) parts cannot be treated as usual theorems about sampling sets due to the necessity of cutting off the vertex of $C(\eta)$ which is unavoidable in view of the examples in [L2]. However, it makes sense to call a set $E$ quasi-sampling if

$$\sup_{x \in C(\eta) \setminus B(0, R(f))} |f(x)| \leqslant C \cdot \sup_{x \in E} |f(x)|, \qquad \forall f \in B(\sigma)$$

for some $R(f) < \infty$.

In this paper we prove the analogues of Theorems B and C for functions analytic in cones of $\mathbf{C}^n$. It is natural to call theorems of this kind Cartwright-type theorems.

It is well known that in many cases the results taking place for entire functions fail to hold for functions analytic on proper subsets of $\mathbf{C}^n$. Even when such results are true, they are technically much harder to prove.

In this paper, we make systematical use of the possibility of a "good" approximation of a function analytic in a cone by entire functions with control of growth. In the case of dimension 1 such an approximation was constructed by M. V. Keldysh [Ke]. For the case of several variables, the result showing the possibility of such an approximation is due to the second author [Ru]. To formulate this result we introduce some notations.

Let $\omega$ and $\varphi$ be plurisubharmonic functions in $\mathbf{C}^n$, both possessing the "non-oscillating" property

$$(u)^{[1]}(z) \leqslant -A(-u)^{[1]}(z) + B,$$

where by $u^{[r]}(z)$ we denote $\sup\{u(w): |z-w|<r\}$, and $A, B \geqslant 0$. Assume also that

$$\varphi(z) \geqslant 0, \qquad \log(1+|z|) = o(\varphi(z)), \qquad |z| \to \infty.$$

For $\varepsilon \geqslant 0$ we denote by $\Omega_\varepsilon$ the set

$$\Omega_\varepsilon = \{z \in \mathbf{C}^n: \omega(z) < -\varepsilon\varphi(z)\}$$

and suppose that

$$\forall \varepsilon_1 > \varepsilon_2: \inf\{|z_1 - z_2|: z_1 \in \Omega_{\varepsilon_1}, z_2 \in \mathbf{C}^n \backslash \Omega_{\varepsilon_2}\} > 0,$$

which is a kind of smoothness condition on $\omega$ and $\varphi$.

THEOREM D [Ru]. *Let $f$ be an analytic function in $\Omega_0$ satisfying the estimate*

$$|f(z)| \leqslant C_f e^{C_f \varphi(z)}, \qquad z \in \Omega_0.$$

*Then for each $\varepsilon > 0$ and each $N \geqslant 1$ there exists such an entire function $g$ that*

$$|f(z) - g(z)| \leqslant C e^{-N\varphi(z)}, \qquad z \in \Omega_\varepsilon,$$
$$|g(z)| \leqslant C e^{C \max(N, C_f) \cdot (2/\varepsilon \cdot \omega^+ + \varphi)(z)}, \qquad z \in \mathbf{C}^n,$$

*where $C$ does not depend on $N$.*

Let $\rho$ be a positive number and let $T_\rho$ be the transformation

$$T_\rho: (z_1, ..., z_n) \mapsto (z_1^{1/\rho}, ..., z_n^{1/\rho}).$$

This transformation obviously maps cones with vertex at the origin onto cones of the same type and maps $\mathbf{R}^n_+$ onto $\mathbf{R}^n_+$.

DEFINITION. Let $E$ and $F$ be subsets of $\mathbf{R}^n_+$, $E$ being measurable. The set $E$ is said to be *dense of order* $\rho$ *relative to* $F$, if $T_\rho^{-1}(E)$ is dense relative to $T_\rho^{-1}(F)$.

DEFINITION. A set $E\subset\mathbf{R}^n_+$ is called an *ε-net of order* $\rho$ for a set $F\subset\mathbf{R}^n$ if $T_\rho^{-1}(E)$ is an ε-net for $T_\rho^{-1}(F)$.

Below we formulate our Cartwright-type theorems for functions holomorphic in cones. Note that we will be able to consider also functions of order $\rho$ different from 1.

For $\rho\geqslant 1$ put $W_\rho=\{z=(z_1, ..., z_n)\in\mathbf{C}^n: \quad -\pi/2\rho<\arg z_j<\pi/2\rho, j=1, ..., n\}$. Note that $W_1=\mathbf{C}^n_+$ ($\mathbf{C}_+$ stands for the right halfplane) and that $Q_\rho\cap\mathbf{R}^n=\mathbf{R}^n_+$ for each $\rho$.

Our first result is

THEOREM 1. *Let $E$ be a dense set of order $\rho\geqslant 1$ relative to $\mathbf{R}^n_+$.*

*Then for every $\eta\in(0, 1)$ each function $f\in H(W_\rho; \rho, \infty)$, which is bounded in a neighborhood of the origin and bounded on $E$ is bounded on $C(\eta)$.*

*Moreover, for each $\sigma\in(0, \infty)$ and $f\in H(W_\rho; \rho, \sigma)$ there exists a positive constant $\Delta=\Delta(E, \rho, \sigma, \eta)$, such that*

$$\sup_{x\in C(\eta)\backslash B(0, R)} |f(x)|\leqslant\Delta\sup_{x\in E} |f(x)|$$

*for some $R=R(f, \eta)<\infty$.*

The corresponding result for ε-nets is

THEOREM 2. *Let $E$ be an ε-net of order $\rho\geqslant 1$ for $\mathbf{R}^n_+$ and let a number $\eta\in(0, 1)$ be given.*

*Then there exists such a number $\sigma_0=\sigma_0(n, E, \rho, \varepsilon, \eta)>0$ that for every $\sigma\in[0, \sigma_0)$, each function $f\in H(W_\rho; \rho, \sigma)$ which is bounded near the origin and bounded on $E$ is bounded on $C(\eta)$.*

*Moreover, if $\sup_{x\in E}|f(x)|>0$, there is a positive constant*

$$\Delta=\Delta(E, \rho, \sigma, \varepsilon, \eta),$$

*such that*

$$\sup_{x\in C(\eta)\backslash B(0, R)} |f(x)|\leqslant\Delta\cdot\sup_{x\in E} |f(x)|$$

*for some $R=R(f, \eta)<\infty$.*

*Remark* 1. The number $\sigma_0$ plays essentially the same role as the number $\pi$ in Cartwright's theorem. The example of the function $f(z) = z * \sin(\pi z)$ shows that the statement of Theorem 2 (for a fixed $\varepsilon$-net) does not hold for all $\sigma$.

*Remark* 2. The (b) parts of Theorems B and C may be reformulated for an arbitrary cone $C \subset \mathbf{R}^n$ with nonempty interior instead of $\mathbf{R}^n_+$, since it is always possible to find such a linear automorphism $\psi: \mathbf{C}^n \to \mathbf{C}^n$, that $\psi(\mathbf{R}^n_+) \subset C$, and to consider $f(\psi(z))$ instead of $f(z)$ (note that the order is not affected). Since the image under $\psi$ of a relatively dense set is a relatively dense set, and the image of an $\varepsilon$-net is an $\varepsilon$-net (possibly, with a different $\varepsilon$), the statements of the theorems require only obvious recalculations of all constants.

*Remark* 3. The (b) part of the Theorem C and respectively Theorem 2 may be strengthened by assuming the set $E$ to be an $\varepsilon$-net not for the whole $\mathbf{R}^n_+$ but for its relatively dense subset.

*Remark* 4. To the best of our knowledge, Theorems 1 and 2 are new in the case $\rho > 1$ even for entire functions.

*Remark* 5. There are examples showing sharpness (in a certain sense) of Theorems B and C; for instance, it was shown [L2] that the results fail to hold if we do not truncate the cone $C(\eta)$ by the ball $B(0, R)$, and that the value of $R$ cannot be chosen independent of $f \in H(1, \sigma)$, etc. It was shown also that the cone $C(\eta)$, a proper subcone of $\mathbf{R}^n_+$, cannot be replaced, for instance, by a translation of $R^n_+$. The same examples with obvious modifications play a similar role for Theorems 1 and 2.

Next we mention V. Bernstein-type theorems for entire functions of finite order. By this we mean results giving conditions on sets sufficient for calculation of the (radial) indicator. Recall that the radial indicator of a function $f \in H(\rho, \infty)$ is defined as

$$h_f(z) = \limsup_{w \to z} \limsup_{t \to \infty} \frac{\log |f(tw)|}{t^\rho}.$$

For the case of dimension 1 the first lim sup (regularization) may be omitted. We refer to [Ro] for the properties of the radial indicator.

V. Bernstein [Ber] was the first to give a sufficient condition on a set $E$ on a ray which guaranties that

$$h_f(1) = \limsup_{t \to \infty,\, t \in E} \frac{\log |f(t)|}{t^\rho}.$$

The references to further results in this direction are given in [L3]. We mention below results of the first author concerning entire functions in $\mathbf{C}^n$.

DEFINITION. Let $\varepsilon(t)$, $t \in \mathbf{R}_+$, be a decreasing function tending to zero as $t \to \infty$. A set $E \subset \mathbf{R}^n$ is called an *$\varepsilon(\ )$-net* for a set $F \subset \mathbf{R}^n$ if for each $x \in F$ there exists $y \in E$ such that

$$|x-y| \leqslant \varepsilon(|x|).$$

DEFINITION. A set $E \subset \mathbf{R}^n_+$ is called an *$\varepsilon(\ )$-net of order $\rho$* for a set $F \subset \mathbf{R}^n_+$ if its preimage under the map $T_\rho$ is an $\varepsilon(\ )$-net for $T_\rho^{-1}(F)$.

THEOREM 4 [L3]. *Let a set $E$ be an $\varepsilon(\ )$-net of order $\rho \in (0, \infty)$ for some cone $C(\eta_0)$.*

*Then the relation*

$$h_f\left(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}\right) = \lim_{\eta \to 0} \limsup_{|x| \to \infty,\, x \in E \cap C(\eta)} \frac{\log|f(x)|}{|x|^\rho}$$

*holds for every function $f \in H(\rho, \infty)$.*

Theorem E yields the following uniqueness result.

THEOREM F [L3]. *Let $E$ be as in Theorem E and let*

$$\limsup_{|x| \to \infty,\, x \in E} \frac{\log|f(x)|}{|x|^\rho} = -\infty$$

*for some function $f \in H(\rho, \infty)$.*

*Then $f \equiv 0$.*

Our Theorem 3 below is an analogue of Theorem E for functions holomorphic in cones.

THEOREM 3. *Let a set $E$ be an $\varepsilon(\ )$-net of order $\rho > 0$ for some cone $C(\eta_0)$.*

*Then the relation*

$$h_f\left(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}\right) = \lim_{\eta \to 0} \limsup_{|x| \to \infty,\, x \in E \cap C(\eta)} \frac{\log|f(x)|}{|x|^\rho}$$

*holds for every function $f \in H(W_\tau; \rho, \infty)$, $\tau \geqslant 1$.*

Note that while the indicator of an entire function of finite type $\sigma$ is bounded below by $-\sigma$ [Ro], the (regularized) indicator of a function holomorphic in a cone needs not to be bounded from below. It can even be identically $-\infty$ for a function which is not identically zero. Hence the corresponding uniqueness result holds only if the cone $W_\tau$, in which our function is defined, is wide enough.

THEOREM 4. *Let $E$ be as in Theorem 3 with $\rho > 1$ and let*

$$\limsup_{|x| \to \infty,\, x \in E} \frac{\log |f(x)|}{|x|^\rho} = -\infty$$

*for some function $f \in H(W_\tau; \rho, \infty)$, $\tau \in (1, \rho)$.*

*Then $f \equiv 0$.*

The idea of most of the proofs in this paper is to approximate a function holomorphic in a cone by an entire function with the help of Theorem D, apply the corresponding theorem for entire functions to the approximating function and derive the required estimates for the initial function. The transparency of such type of argument should not be confused, however, with the "triviality of results." The main gain in the "compilation" of two kinds of our known theorems is that we are able to obtain new results, and not only for functions in cones but also for entire functions. For another application of the approximation techniques similar to Theorem D, see [Ru1].

## 2. SOME REMARKS CONCERNING CONES IN $\mathbf{C}^N$

In this paper we will deal mainly with two types of cones in $\mathbf{C}^n$.

One of them, $W_\tau$, is defined in the previous section. We introduce another one.

For $t > 0$ denote by $\|\cdot\|_t$ a norm in $\mathbf{C}^n$ given by

$$\|z\|_t = \max_{j=1, \dots, n} \left\{ |\operatorname{Re} z_j|, \left|\operatorname{Im} \frac{z_j}{t}\right| \right\}.$$

By $Y_t(\eta)$, $\eta \in [0, 1)$, we denote the cone in $\mathbf{C}^n$ given by

$$Y_t(\eta) = \{ z \in \mathbf{C}^n_+ : \min_{j=1, \dots, n} \operatorname{Re} z_j > \eta \, \|z\|_t \}.$$

Note that for all $t > 0$ the intersection of $Y_t(\eta) \cap \mathbf{R}^n_+$ is exactly the real cone $C(\eta)$. Obviously, $U_t(0) = \mathbf{C}^n_+$.

The geometry of the cone $Y_t(\eta)$ is very simple. We just observe that the ray

$$l = \{ \xi(1, \dots, 1), \xi > 0 \}$$

lies on the complex line

$$\mathscr{L} = \{z_1 = \cdots = z_n\}$$

which has the largest intersection with $Y_t(\eta)$:

$$\mathscr{L} \cap Y_t(\eta) = \left\{ w(1, ..., 1) : w \in \mathbf{C},\ |\arg w| < \arctan \frac{t}{\eta} \right\}.$$

One easily sees that, given a number $\tau \geqslant 1$, it is possible to choose such $t$ and $\eta$ that

$$Y_t(\eta) \subset W_\tau$$

and

$$\mathscr{L} \cap Y_t(\eta) = \mathscr{L} \cap W_\tau.$$

We would like to write each of the two types of cones in the form $\{z \in \mathbf{C}^n : u(z) < 0\}$ for some plurisubharmonic function $u$ in $\mathbf{C}^n$. In both cases we can take $u$ to be of order 1:

$$u(z) = \max_{j=1, ..., n} (-\operatorname{Re} z_j) + \eta \, \|z\|_t$$

for $Y_t(\eta)$,

$$u(z) = \max_{j=1, ..., n} \left\{ |\operatorname{Im} z_j| - \tan \frac{\pi}{2\tau} \cdot \operatorname{Re} z_j \right\}$$

for $W_\tau$.

## 3. PROOF OF THEOREMS 1 AND 2

*Proof of Theorem* 1. The idea of the proof is to approximate the function $f$ by an entire function $g$ with the help of Theorem D, apply Theorem B(b) to $g$ and derive the required estimates for $f$.

Let $T_\rho$ be the transformation mentioned in the Introduction. If $f$ is holomorphic in $W_\rho$ and has order $\rho$, then $f(T_\rho(z))$ is holomorphic in $W_1 = \mathbf{C}^n_+$ and has order 1.

Thus, in view of the mentioned properties of the transformation $T_\rho$, it is enough to assume $\rho = 1$ in what follows.

Put

$$F = \sup_{x \in E} |f(x)|.$$

If $F = 0$ then $f \equiv 0$; if $F = \infty$ then there is nothing to prove. Therefore we can assume that $F \in (0, \infty)$ and normalize $f(z)$ so that $F = e^{-1}$.

Next we define functions $\omega$ and $\varphi$ in the following way. Put

$$\omega = \max_{j=1, \ldots, n} (-\operatorname{Re} z_j), \qquad \varphi = \max(1, \|z\|_t),$$

where $t > 0$ is arbitrary.

Then the set

$$\Omega_0 = \{z : \omega(z) < 0\}$$

is exactly $\mathbf{C}^n_+$, and for each $\eta \in (0, 1)$ the set

$$\Omega_\eta = \{z : \omega(z) < -\eta\varphi(z)\}$$

coincides with $Y_t(\eta)$ without some neighborhood of the vertex. Besides that,

$$\Omega_\eta \cap \mathbf{R}^n_+ \supset C(\eta) \backslash B(0, \eta).$$

It is clear that the conditions of Theorem D are satisfied (we can assume that $|f| \leqslant C_f e^{C_f \varphi}$ everywhere in $\Omega_0$, otherwise replace $\Omega_0$ with $\Omega_\delta$ for some small $\delta < \eta/4$, see also Remark 2 following Theorem 2).

Let $f(z)$ be a given function from the class $H(W_1; 1, \sigma)$. By Theorem D, there exists an entire function $g(z)$ of exponential type $\leqslant K\sigma$, with $K = K(n, \eta)$ not depending on $f \in H(W_1; 1, \sigma)$, such that

$$|f(z) - g(z)| \leqslant e^{-\varphi(z)} \leqslant F, \qquad z \in \Omega_{\eta/2}.$$

Therefore

$$\sup_{x \in E \cap C(\eta/2) \backslash B(0, \eta/2)} |g(x)| \leqslant 2F.$$

Since

$$E \cap C(\eta/2) \backslash B(0, \eta/2)$$

is dense (of order 1) relative to $C(\eta/2)$, Remark 2 following Theorem 2 implies that

$$\sup_{x \in C(\eta)} |g(x)| < \infty,$$

and for some $\Delta < \infty$ and $R < \infty$.

$$\sup_{x \in C(\eta) \setminus B(0, R)} |g(x)| \leqslant \Delta \cdot \sup_{x \in E \cap C(\eta/2) \setminus B(0, \eta/2)} |g(x)| \leqslant 2\Delta F.$$

Since $|f(x) - g(x)| \leqslant F$ on $\Omega_{\eta/2}$, we obtain the required estimate for our function $f$.

The theorem is proved.

*Proof of Theorem* 2. The proof repeats the proof of the previous theorem with the only difference that Theorem C(b) is applied instead of Theorem B(b). Note that if $E$ was an $\varepsilon$-net for $\mathbf{R}^n_+$, then $E \cap C(\eta/2) \setminus B(0, \eta/2)$ would be a $4\varepsilon$-net for $C(\eta/2)$ for any $\eta \in (0, \varepsilon)$.

## 4. PROOF OF THEOREMS 3 AND 4

*Proof of Theorem* 3. Given a function $f$ analytic in $W(\tau)$ and of order $\rho$, denote $h_f(1/\sqrt{n}, ..., 1/\sqrt{n})$ by $H_f$ and let $H_f(E)$ be the corresponding limit calculated over the set $E$. It is obvious that

$$H_f(E) \leqslant H_f$$

We need to prove the converse.

The way to do this is to use Theorem D to find an entire function $g \in H(\rho, \infty)$ with the properties

$$H_g = H_f$$

and

$$H_g(E) = H_f(E)$$

and use Theorem E for entire functions to prove that

$$H_g(E) = H_g,$$

which yields the desired relation.

First, we can assume that $C(\eta) = \mathbf{R}^n_+$ (otherwise apply a linear transformation $\psi$ which does not change the order and leaves our set $E$ and $\varepsilon(R)$-net of the same order). Next, as in the previous section, we can assume that $\rho = 1$ (apply the transformation $T_\rho$ otherwise; $E$ becomes then an $\varepsilon(R)$-net of order 1 and $W_\tau$ becomes $W_{\tau'}$ for some $\tau'$ which can be assumed $\geqslant 1$).

We consider two cases. Assume first that $H_f(E) > -\infty$. Since the multiplication of our function by $e^{A(z_1 + \cdots + z_n)}$ does not affect the investigated property of the set $E$, we can always assume that $H_f(E) > 0$. Hence an entire function $g$ uniformly approximating $f$ in a cone containing the ray $\ell = \{(t, ..., t), t > 0\}$ will have the same values of $H_g(E)$ and $H_g$ as $f$. Thus, in view of Theorem E, it is enough to construct such a function.

Take $N = 1$, $\omega(z) = \max_{j=1,...,n}(-\operatorname{Re} z_j) + \eta' \|z\|_t$, for such $t$ and $\eta'$ that the cone $\Omega_0 = \{z = \omega(z) < 0\}$ is contained in $W_\tau$, and take $\varphi(z) = \max(\delta, \|z\|_t)$, $\delta > 0$. For $\varepsilon \in (0, 1 - \eta')$ the set $\Omega_\varepsilon = \{z = \omega(z) < -\varepsilon\varphi(z)\}$ which is $Y_t(\eta' + \varepsilon)$ without a neighborhood of the vertex) has the property

$$\Omega_\varepsilon \cap \mathbf{R}^n_+ \supset C(\eta' + \varepsilon) \backslash B(0, \varepsilon\delta).$$

Applying Theorem D, we are done.

Now consider the case $H_f(E) = -\infty$. We need to prove that $H_f = -\infty$. We choose $\omega$ and $\varphi$ to be the same as above. Fix some $\varepsilon \in (0, 1 - \eta)$ and denote by $g_N$ the entire function of finite type corresponding to the choice of $N \geqslant 1$ in Theorem D. Note that for any such function

$$H_{g_N}(E) = H_{g_N}.$$

The entire function $g_N$ satisfies $|f(z) - g_N(z)| < e^{-N\|z\|_t}$ on $\Omega_\varepsilon$, in particular, on $C(\varepsilon')$ for $|x|$ large enough. We have

$$\begin{aligned} H_{g_N} &\leqslant \lim_{\eta \to} \limsup_{|x| \to \infty,\, x \in E \cap C(\eta)} \frac{\log(|f(x)| + |g_N(x) - f(x)|}{|x|} \\ &\leqslant \max(-N, H_f(E)) \\ &= -N, \end{aligned}$$

and

$$\begin{aligned} H_f &\leqslant \lim_{\eta \to 0} \lim_{|x| \to \infty,\, x \in C(\eta)} \frac{\log(|g_N(x)| + |g_N(x) - f(x)|)}{|x|} \\ &= \max(-N, N_{g_N}) \\ &= -N. \end{aligned}$$

Since $N$ was arbitrary, we conclude that $H_f = -\infty$. The theorem is proved.

*Remark.* In the case $\rho \geqslant 1$ it is also possible to give another proof of Theorem 3 based on Theorem 2 (and thus using Theorem D indirectly).

*Proof of Theorem* 4. Let $\ell = \{(t, ..., t), t > 0\}$. The conditions of Theorem 4 imply that the rays $e^{i(\pi/2\rho)}\ell$ and $e^{-i(\pi/2\rho)}\ell$ both belong to the cone where our function is holomorphic. From Theorem 3 it follows then that the plurisubharmonic function $H_f(z) = -\infty$ on some cone $C(\eta) \subset R^n_+$.

Since $C(\eta)$ is a non-pluripolar set, $H_f(z) = -\infty$ where it is defined, in particular, on the two rays mentioned above. By the properties of the indicator [Ro, Ch. 3, section 5], $f \equiv 0$.

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